JOURNAL OF COMBINATORIAL THEORY, Series B 43, 60–69 (1987)

On the Euler Genus of a 2-Connected Graph

R. BRUCE RICHTER

Department of Mathematics,
Utah State University, Logan, Utah 84322

Communicated by the Editors

Received December 17, 1984

The *Euler genus* of the surface Σ obtained from the sphere by the addition of k crosscaps and h handles is $\varepsilon(\Sigma) = k + 2h$. For a graph G, the *Euler genus* $\varepsilon(G)$ of G is the smallest Euler genus among all surfaces in which G embeds. The following additivity theorem is proved.

THEOREM. *Suppose* $G = H \cup K$, where H *and* K *have exactly the vertices* v *and* w *in common. Then* $\varepsilon(G) = \min\{\varepsilon(H + vw) + \varepsilon(K + vw),\ \varepsilon(H) + \varepsilon(K) + 2)$.

1. INTRODUCTION

If the surface Σ is obtained from the sphere by the addition of h handles and k crosscaps, then the *Euler genus* $\varepsilon(\Sigma)$ is defined to be $k + 2h$. If the surface Σ has connected components $\Sigma_1, ..., \Sigma_k$, then define $\varepsilon(\Sigma)$ to be $\varepsilon(\Sigma_1) + \cdots + \varepsilon(\Sigma_k)$. For a graph G, the *Euler genus* $\varepsilon(G)$ of G is the least element in the set $\{\varepsilon(\Sigma) \mid G \text{ embeds in } \Sigma)$.

The purpose of this paper is to prove the following "additivity" result.

THEOREM 1. *Let* H_1 *and* H_2 *be connected graphs such that* $H_1 \cap H_2$ *consists of the two isolated vertices* v *and* w. *Then*:

$$\varepsilon(H_1 \cup H_2) = \min\{\varepsilon(H_1 + vw) + \varepsilon(H_2 + vw),\ \varepsilon(H_1) + \varepsilon(H_2) + 2). \quad \text{(A)}$$

Decker [5] and Decker *et al.* [7] have established an orientable analogue of Theorem 1; we shall discuss this more fully in Section 5. Miller [10] has shown that $\varepsilon(H_1 \cup H_2) \geqslant \varepsilon(H_1) + \varepsilon(H_2)$, from which it follows that $\varepsilon((H_1 \cup H_2) + vw) = \varepsilon(H_1 + vw) + \varepsilon(H_2 + vw)$, which is a special case of our result. As it makes our work somewhat shorter, we shall make use of Miller's result.

Our broad outline is quite similar to the arguments used in [3]. In particular, we show that the minimum in Eq. (A) is both an upper and lower

0095-8956/87 $3.00

bound for $\varepsilon(H_1 \cup H_2)$. The upper bound is quite easy and some form of it has been known for years (see [1, 8, 13]).

The lower bound is more difficult. We are required to split an embedding of $H_1 \cup H_2$ into appropriate embeddings of H_1 and H_2; this is the bulk of this work.

The Upper and Lower Bound Lemmas are stated in Section 2 and Theorem 1 is proved using them. In Section 3 is a proof of the Lower Bound Lemma. A discussion of the orientable analogue to Theorem 1 is givn in Section 4.

For this work, it is assumed the reader is familiar with embeddings of graphs in surfaces. We do not assume that our embeddings are 2-cell embeddings. This extra generality is useful in [12], in which the non-orientable analogue of Theorem 1 is proved.

The methods used here will follow those of [9] and [15], rather than the algebraic techniques of [13]. Much of the technical justification for what is done here can be found in [11].

To illustrate the relevant ideas, consider the graph G embedded in the torus in Fig. 1. The face F has $(a', d, b, a, c', b', a')$ as the vertex-sequence of its *boundary walk*; this is the closed walk in G obtained by once traversing the perimeter of G. Note that a boundary walk need not be a circuit , although it does induce a connected subgraph of G. Because we allow faces to be other than discs, it may be the case that one face has several distinct boundary walks in its closure.

If H is the subgraph of G induced by the vertices $\{a, b, c, d\}$, then the *induced embedding* of H in the torus is given in Fig. 2; the vertices and edges of H are embedded as they are in the original embedding of G. The boundary walk $P = (a, b, d, a, c, d, b, c, a)$ of F' is not a circuit.

Relabel P as $(v_0, v_1, \ldots, v_8)$, so $v_0 = a$, $v_1 = b$, etc. Then $v_0 = a = v_3$. As v_3,

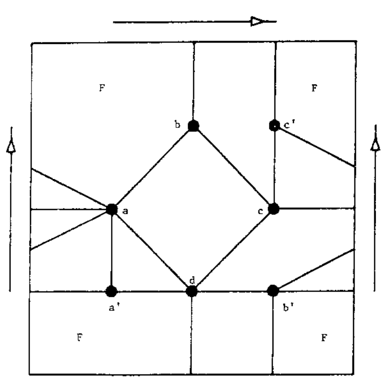

FIGURE 1

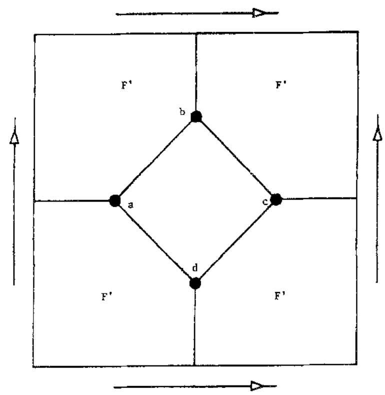

FIGURE 2

a is incident with two edges in $E(G)-E(H)$ that lie in F'; as v_0, a is incident with only one such edge. The *relative degree* of v_3 in P is defined to be 2, while v_0 has relative degree 1 in P. The reader can verify that v_2, v_5, and v_7 each has relative degree 1 in P.

Finally, we note that there is a neighbourhood of P in F' that looks like the square annulus illustrated in Fig. 3. The line segments drawn represent portions of edges in $E(G)-E(H)$ that lie in F'.

2. PROOF OF THEOREM 1

In this section, the Upper Bound and Lower Bound Lemmas are stated and used to prove Theorem 1.

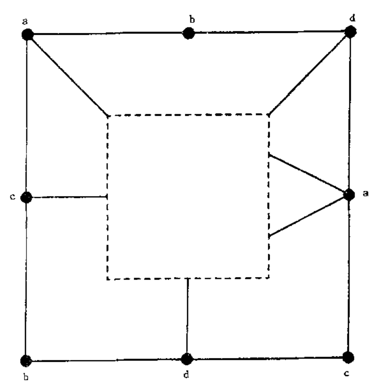

FIGURE 3

Upper Bound Lemma. *Let H_1 and H_2 be graphs such that $H_1 \cap H_2$ consists of the isolated vertices v and w. Then*:

(i) *For $j=1, 2$ if g_j: $H_j + vw \to \Sigma_j$ is an embedding, then there is an embedding g: $(H_1 \cup H_2) + vw \to \Sigma$ such that $\varepsilon(\Sigma) = \varepsilon(\Sigma_1) + \varepsilon(\Sigma_2)$.*

(ii) *For $j=1, 2$, if g_j: $H_j \to \Sigma_j$ is an embedding, then there is an embedding g: $(H_1 \cup H_2) \to \Sigma$ such that $\varepsilon(\Sigma) = \varepsilon(\Sigma_1) + \varepsilon(\Sigma_2) + 2$.*

This result is straightforward, so we omit its proof. It really describes how to join embeddings of H_1 and H_2 to obtain an embedding of $H_1 \cup H_2$. The Lower Bound Lemma reverses the process.

Lower Bound Lemma. *Let H_1 and H_2 be connected graphs with $H_1 \cap H_2$ consisting of the isolated vertices v and w. Let g: $H_1 \cup H_2 \to \Sigma$ be an embedding and let g_1 be the induced embedding of H_1. Then*:

(i) *if g_1 has a boundary walk containing both v and w, then, for $j=1, 2$, there is an embedding h_j: $H_j + vw \to \Sigma_j$ such that $\varepsilon(\Sigma) = \varepsilon(\Sigma_1) + \varepsilon(\Sigma_2)$.*

(ii) *if g_1 has no boundary walk containing both v and w, then, for $j=1, 2$, there is an embedding h_j: $H_j \to \Sigma_j$ such that $\varepsilon(\Sigma) = \varepsilon(\Sigma_1) + \varepsilon(\Sigma_2) + 2$.*

We shall not prove both parts of the Lower Bound Lemma here. Part (i) is, in essence, a rewording of Miller's result, which we shall quote in the relevant place in the proof of Theorem 1. A refinement of (i) is required in the proof of the non-orientable analogue of Theorem 1; the proof of this refinement is given in [12]. Thus, this paper combines with [12] to give a proof of Theorem 1 that is completely independent of Miller's work.

Part (ii) of the Lower Bound Lemma is new and is the main contribution of this paper; it is proved in Section 3. The motivation for presenting the Lower Bound Lemma as we have here is to emphasize the symmetry between it and the Upper Bound Lemma.

Proof of Theorem 1. That $\varepsilon(H_1 \cup H_2) \leqslant \min\{\varepsilon(H_1 + vw) + \varepsilon(H_2 + vw), \varepsilon(H_1) + \varepsilon(H_2) + 2\}$ is an immediate consequence of the Upper Bound Lemma.

To establish the lower bound, it suffices to prove either $\varepsilon(H_1 \cup H_2) \geqslant \varepsilon(H_1 + vw) + \varepsilon(H_2 + vw)$ or $\varepsilon(H_1 \cup H_2) \geqslant \varepsilon(H_1) + \varepsilon(H_2) + 2$. To this end, let g: $H_1 \cup H_2 \to \Sigma$ be an embedding such that $\varepsilon(\Sigma) = \varepsilon(H_1 \cup H_2)$. Let g_1 denote the induced embedding of H_1 in Σ. We consider two cases.

Case 1. There is a boundary walk of g_1 that contains both v and w. In this case, we can obtain an embedding g': $(H_1 \cup H_2) + vw \to \Sigma$. Evidently, $\varepsilon((H_1 \cup H_2) + vw) = \varepsilon(\Sigma)$. By Miller's result, $\varepsilon((H_1 \cup H_2) + vw) \geqslant$

$\varepsilon(H_1+vw)+\varepsilon(H_2+vw)$, so $\varepsilon(H_1 \cup H_2) \geqslant \varepsilon(H_1+vw)+\varepsilon(H_2+vw)$, as required.

Case 2. There is no boundary walk of g_1 that contains both v and w. Then, by (ii) of the Lower Bound Lemma, there are embeddings h_j: $H_j \to \Sigma_j$, for $j=1, 2$, such that $\varepsilon(\Sigma)=\varepsilon(\Sigma_1)+\varepsilon(\Sigma_2)+2$. Evidently, $\varepsilon(\Sigma_j) \geqslant \varepsilon(H_j)$, for $j=1, 2$, so $\varepsilon(H_1 \cup H_2)=\varepsilon(\Sigma) \geqslant \varepsilon(H_1)+\varepsilon(H_2)+2$. ∎

The proof of Theorem 1 was broken into two cases based on the way the Lower Bound Lemma was stated. We could just as easily considered the induced embedding of H_2, or the embedding g.

3. Lower Bound Lemma

The proof of Theorem 1 will be completed with the proof of (ii) of the Lower Bound Lemma.

Proof of Lower Bound Lemma (ii). Let g: $H_1 \cup H_2 \to \Sigma$ be an embedding such that v and w occur together in no boundary walk of g_1, the induced embedding of H_1 in Σ. Since H_2 is connected, there is an arc L in H_2 joining v and w. Let g' be the induced embedding of $H_1 \cup L$. For ease of notation, we shall refer to L as vw when discussing g'.

Our first goal is to obtain the embedding g^1: $H_1 \to \Sigma^1$. There will be two occurrences of vw in the boundary walks of g'. They must occur as $(..., v, vw, w,..., w, wv, v,...)$, since otherwise a boundary walk of g_1 will contain both v and w, contradicting the hypothesis. Thus, this is the only boundary walk of g' that contains both v and w. (All other boundary walks of g' are also boundary walks of g_1.)

As shown in (15), we can replace each face of g' with a collection of disc homeomorphs to obtain the embedding g'': $H_1 \cup L \to \Sigma''$. The boundary walks of this embedding are precisely those of g'. (Recall that we do not require our embeddings to be 2-cell. Also, boundary walks induce connected subgraphs, even though a face may have more than one boundary walk.) Let P be the boundary walk of g' that contains vw $(=L)$; as noted earlier, this edge is traversed once in each direction in P. Therefore, the removal of vw creates a face with a handle. As H_1 is connected, the handle may be removed and capped with discs. This yields an embedding g^1: $H_1 \to \Sigma^1$ satisfying

$$\varepsilon(\Sigma^1)=\varepsilon(\Sigma'')-2. \tag{B}$$

We are now interested in obtaining an embedding of H_2. To do this, we note that, except for L, H_2 is embedded in the faces of g'. We show how to glue these faces to a sphere with holes in order to construct an embedding of H_2.

Let $P_1, \ldots, P_q$ be the boundary walks of g', ordered so that $P_r = P$ is the boundary walk containing L $(= vw)$ and $P_1, \ldots, P_r$ are all the boundary walks of g' that contain v. Since g_1 has no boundary walk containing both v and w and $P_1, \ldots, P_{r-1}$, $P_{r+1}, \ldots, P_q$ are also boundary walks of g_1, it follows that if w is a vertex term of P_j, then $j \geqslant r$.

If $j < r$ and v_k is a vertex term of P_j having positive relative degree, then $v_k = v$; for v_k is a vertex of H_1 and is incident with an edge of H_2. Similarly, if $j > r$ and v_k is a vertex term of P_j having positive relative degree, then $v_k = w$. If v_k is a vertex term of P_r having positive relative degree, then v_k is a vertex of L.

For $j = 1, \ldots, q$, let D_j be a square annulus representing a neighbourhood of P_j, as in Fig. 4. For $j \leqslant r$, we ensure that each vertex term of P_j equal to v is represented by a point on the top edge of D_j. For $j \geqslant r$, we require that the representatives of w are on the bottom edge of D_j.

For each boundary walk P_j of g', let C_j denote the cycle in Σ which corresponds to the "inside" (i.e., dashed square in Fig. 4) of D_j. We now cut Σ along each of the cycles C_j, $1 \leqslant j \leqslant q$. Now cap each of the 2 copies of C_j with discs. The resulting manifold has one component which contains the connected graph $H_1 \cup L$ and is homeomorphic to Σ''. There are s other components, where s is the number of faces of g'. We call the s components $\Sigma^\sim$. We note that $\Sigma^\sim$ contains copies of the cycles C_j and these bound discs in $\Sigma^\sim$. Also, by Euler's formula, we have

$$\varepsilon(\Sigma) = \varepsilon(\Sigma'') + \varepsilon(\Sigma^\sim) + 2(q - s). \tag{C}$$

We now describe the surface Σ^2. Consider the sphere and the embedding of L into the sphere shown in Fig. 5. In this figure, there are q dashed squares, which are to correspond to the cycles C_j, $j = 1, \ldots, q$. We form Σ^2 by deleting from $\Sigma^\sim$ the discs bounded by the C_j's, deleting from the sphere the discs bounded by the dashed squares, and then identifying each C_j in $\Sigma^\sim$ with its corresponding square in the sphere. We require that these indentifications match the top, bottom and sides, respectively, of the

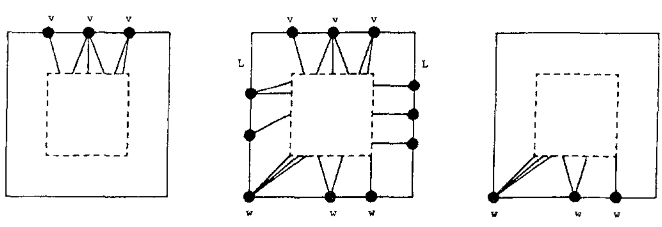

FIGURE 4

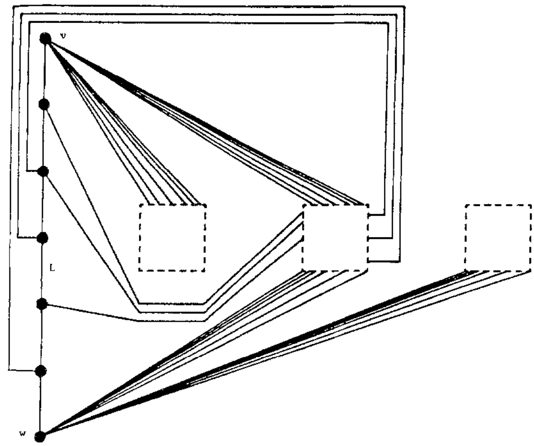

FIGURE 5

squares in D_j and with the squares in Fig. 5. This construction thus forms an embedding of H_2 in Σ^2. By Euler's formula, we have

$$\varepsilon(\Sigma^2) = \varepsilon(\Sigma_0) + \varepsilon(\Sigma^{\sim}) + 2(q - s), \tag{D}$$

where Σ_0 denotes the sphere.

As $\varepsilon(\Sigma_0) = 0$, Eqs. (B), (C), and (D) yield $\varepsilon(\Sigma) = \varepsilon(\Sigma^1) + \varepsilon(\Sigma^2) + 2$, as required. ∎

4. THE ORIENTABLE CASE

The orientable analogue of Theorem 1 has been established by Decker [5]. See also [6, 7, 13]. In this section, we show how to prove it with analogues of the results of Sections 2 and 3. To begin, we need the important notion of a "weave."

Consider the embedding of $K_{3,3}$ in the torus illustrated in Fig. 6. There is exactly one boundary walk of this embedding that is of the form $(v, vw, w,..., v,..., w,...)$. In this walk, v and w are "woven" together. We remark that if a boundary walk has such a weave, then the construction required to prove (i) of the Lower Bound Lemma does not apply (this construction is essentially the same as that of [2] and [14]). We need the existence of a "weave-free" embedding. In the orientable case, such an embedding need not exist.

Specifically, let G be a graph and let x and y be distinct vertices of G. Let $P = (v_0, e_1,..., e_n, v_n)$ be a closed walk of G. A subwalk $Q = (v_j, e_{j+1},..., e_k, v_k)$ of P is an (x, y)-*weave* if there are integers n and p such that:

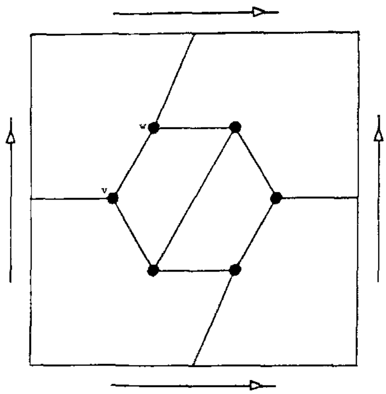

FIGURE 6

(i) $j<n<p<k$; (ii) $x=v_j=v_p$: (iii) $y=v_n=v_k$; and (iv) no proper subwalk of Q satisfies (i)–(iii) for some integers n and p.

The (x, y)-*weave number* of P, denoted $W(P, (x, y))$, is defined to be 0 if P contains no (x, y)-weave, and r if P can be written in the form $(x, A_0, y, B_0, x, A_1, y, \ldots, y, B_r, x)$, where, for $j=0, 1, \ldots, r-1$, $(x, A_j, y, B_j, x, A_{j+1}, y)$ is an (x, y)-weave, and (x, A_r, y, B_r, x) contains no (x, y)-weave. Note that the weave-number is one less than the alternation number of [5–7].

For an embedding $g\colon G \to \Sigma$, define $W(g, (x, y))$ to be $\Sigma_p W(P, (x, y))$, where the sum is over all boundary walks P of g. Details of how to get (i) of the Lower Bound Lemma are given in [12]. The main point is to get an embedding of $H_1 \cup H_2$ such that the induced embedding of H_1 has no (v, w)-weave. Although not stated in these terms, this is the bulk of Miller's work [10], as well.

In the orientable case, however, it need not be the case that there is an appropriate embedding of $H_1 \cup H_2$ that is weave-free. However, we can prove the following (details are omitted here, see [6, 7]).

For the orientable surface Σ, let $\gamma(\Sigma)$ denote the number h of handles such that Σ is homeomorphic to the sphere with h handles. For a graph G, define the *genus* $\gamma(G)$ to be the least number in $\{\gamma(\Sigma) \mid G \text{ embeds in } \Sigma\}$.

WEAVE LEMMA [7, Corollary 1.5]. *Let G be any graph and let x and y be vertices of G. Let $W(G, (x, y)) = \max\{W(g, (x, y)) \mid g\colon G \to \Sigma$ and $\gamma(\Sigma) = \gamma(G)\}$. Then $W(G, (x, y))$ is either* 0 or 1.

The statement of the orientable analogue of Theorem 1 involves the number $W(G, (v, w))$.

THEOREM 2 [7, Theorem 0.1]. *Let H_1 and H_2 be connected graphs* *having exactly the vertices* v *and* w *in common. For* $j=1, 2$, *let* $\sigma_j = \gamma(H_j+vw)-\gamma(H_j)$ *and let* $W_j = W(H_j+vw, (v, w))$. *Then*

$$\gamma(H_1 \cup H_2) = \gamma(H_1) + \gamma(H_2) + \min\{1, \sigma_1 + \sigma_2 - W_1 W_2\}.$$

The basic outline for the proof of this result is essentially the same as for Theorem 1. In the Upper Bound Lemma, we have the same constructions, plus a new one.

UPPER BOUND EXTENSION. *For* $j=1, 2$, *let* g_j: $H_j \to \Sigma$ *be an embedding such that* $W(g_j, (v, w)) = 1$. *Then there is an embedding* g: $H_1 \cup H_2 \to \Sigma$ *such that* $\gamma(\Sigma) = \gamma(\Sigma_1) + \gamma(\Sigma_2) - 1$.

For the Lower Bound, the proof of the Weave Lemma can be adapted to prove the

WEAVE LEMMA #2. *Let* g: $H_1 \cup H_2 \to \Sigma$ *be an embedding, with* Σ *orientable. Then there is an embedding* g': $H_1 \cup H_2 \to \Sigma$ *such that, if* g'' *is the embedding of* H_1 *induced by* g', *then* $W(g'', (v, w)) \leqslant 1$.

The second statement of the orientable lower bound in analogous to (ii) of the Lower Bound Lemma. However, (i) requires modification.

ORIENTABLE LOWER BOUND. *Let* H_1 *and* H_2 *be connected graphs having precisely the vertices* v *and* w *in common. Let* g: $H_1 \cup H_2 \to \Sigma$ *be an embedding, with* Σ *orientable. By the Weave Lemma #2, we may assume* $W(g_1, (v, w)) \leqslant 1$, *where* g_1 *is the induced embedding of* H_1. *Then*:

(i) *if some boundary walk of* g_1 *contains both* v *and* w, *then, for* $j=1, 2$, *there is an embedding* h_j: $H_j + vw \to \Sigma_j$ *such that* $\gamma(\Sigma) + W(g_1, (v, w)) = \gamma(\Sigma_1) + \gamma(\Sigma_2)$.

(ii) *if no boundary walk of* g_1 *contains both* v *and* w, *then, for* $j=1, 2$, *there is an embedding* h_j: $H_j \to \Sigma_j$ *such that* $\gamma(\Sigma) = \gamma(\Sigma_1) + \gamma(\Sigma_2) + 1$.

REFERENCES

1. S. ALPERT, The genera of amalgamations of graphs, *Trans. Amer. Math. Soc.* **78** (1973), 1–39.
2. D. ARCHDEACON AND P. HUNEKE, On irreducible graphs for non-orientable surfaces, preprint.
3. J. BATTLE, F. HARARY, Y. KODAMA, AND J. W. T. YOUNGS, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.* **68** (1962), 565–568.
4. M. BROWN, Locally flat imbeddings of topological manifolds, *Ann. of Math.* **75** (1962), 331–341.
5. R. DECKER, Ph.D. dissertation, The Ohio State University, Columbus, 1978.

6. R. DECKER, H. GLOVER, AND P. HUNEKE, The genus of 2-amalgamations of graphs, *J. Graph Theory* **5** (1981), 95–102.
7. R. DECKER, H. GLOVER, AND P. HUNEKE, Computing the genus of the 2-amalgamations of graphs, *Combinatorica* **5** (1985), 271–282.
8. F. HARARY AND Y. KODAMA, On the genus of an n-connected graph, *Fund. Math.* **54** (1964), 7–13.
9. P. HOFFMAN AND B. RICHTER, Embedding graphs in surfaces, *J. Combin. Theory Ser. B* **36** (1984), 65–84.
10. G. MILLER, An additivity theorem for the genus of a graph, preprint.
11. R. B. RICHTER, Ph.D. dissertation, University of Waterloo, 1983.
12. R. B. RICHTER, On the non-orientable genus of a 2-connected graph, *J. Combin. Theory Ser. B* **43** (1987), 48–59.
13. S. STAHL, Permutation-partition pairs: a combinatorial generalization of graph embeddings, *Trans. Amer. Math. Soc.* **159** (1980), 129–145.
14. S. STAHL AND L. BEINEKE, Blocks and the non-orientable genus of a graph, *J. Graph Theory* **1** (1977), 75–78.
15. J. W. T. YOUNGS, Minimal imbeddings and the genus of a graph, *J. Math. Mech.* **12** (1963), 303–315.